INTERNAL WAVES OF FINITE AMPLITUDE

R. L. Kulyaev

UDC 532.501.32

Much research has been devoted to unsteady fluid flow with a free boundary. For example, Ovsyannikov [1] and Nalimov [2] have proven theorems on the existence and uniqueness of a solution, and a number of papers have proposed algorithms for numerical solution, based on various chain methods [3-6] or potential-theory methods [7-9]. In the present article we consider two-dimensional potential waves of finite amplitude on the interface between two heavy fluids of different densities. The initial problem is reduced to the Cauchy problem for a system of two integrodifferential equations. An algorithm for the numerical solution of this system is constructed, and the results of calculations are presented.

1. We consider the plane motion of two nonviscous, incompressible fluids of different densities in the field of gravity. The flow is assumed to be continuous in the whole plane, potential flow outside the interface line separating the fluids and periodic in the horizontal direction.

Let a Cartesian coordinate system x, y move in the horizontal direction with velocity equal to one half the sum of the flow velocities infinitely far from the interface line L, and let the y axis be directed vertically upwards (Fig. 1). In the upper (D_1) and lower (D_2) flow domains the fluid velocity $V = (V_x, V_y)$ satisfies the equations

$$\operatorname{div}\mathbf{V}=0,\ \operatorname{curl}\mathbf{V}=0,\ (x,\ y)\in D_n,\ n=1,\ 2 \tag{1.1}$$

and the following boundary conditions: The perturbed flow velocity damps out as we become removed from the interface line

$$\mathbf{V}(x, y, t)\to\begin{cases}(-v_\infty, 0), & y\to+\infty\\ (v_\infty, 0) & y\to-\infty\end{cases} \tag{1.2}$$

no fluid flows across the interface line

$$\mathbf{v}_n\cdot\mathbf{\nu}=\mathbf{v}\cdot\mathbf{\nu},\ n=1,\ 2, \tag{1.3}$$

and the drop in the hydrodynamic pressure at the interface line obeys the Laplace law

$$p_1-p_2=\mu k. \tag{1.4}$$

Here t is time, v_∞ = const, $\boldsymbol{\nu}$ is a unit normal to L, $\mathbf{v}$ is the translation velocity of the line L, $\mathbf{v}_n$ and p_n are the limiting values of the velocity V and pressure p, respectively, on approaching L from the domain D_n, μ is the coefficient of surface tension, and k is the curvature of the interface line, with $k<0$ ($k>0$) if the domain D_2 is convex (concave) in the neighborhood of the point in question. The initial velocity field

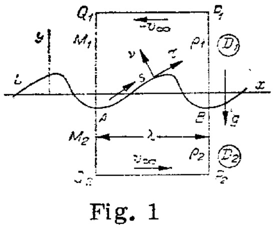

Fig. 1

$$\mathbf{V}(x,\ y,\ 0)=\mathbf{V}_0(x,\ y) \tag{1.5}$$

is assumed to be known and to satisfy conditions (1.1)-(1.3).

Insofar as the interface line L(t) is not known beforehand, the problem as stated is nonlinear.

2. Let us derive the equations of motion of the wave surface L, assuming that the surface has no self-intersection points and that as functions of

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 1, pp. 96-105, January-February, 1975. Original article submitted April 29, 1974.

arc length s the coordinates of the wave surface and the velocity discontinuity $v_1 - v_2$ are continuously differentiable to some order (the required order of smoothness will be made more precise below in Sec. 4).

By virtue of (1.3) the discontinuity in the flow velocity at the line L satisfies the equation

$$v_2 - v_1 = \gamma \partial\zeta/\partial s, \tag{2.1}$$

where the function γ is real, and $\zeta = \xi + i\eta$ is the complex coordinate of a point on the interface line. The solution of the associated Riemann boundary-value problem [10] enables us to represent the velocity field in the form

$$\overline{V}(z, t) = \frac{1}{2\lambda i}\int_0^{l(t)} \gamma(s, t)\,\mathrm{ctg}\frac{\pi}{\lambda}\{z - \zeta(s, t)\}\,ds, \quad z \in L, \tag{2.2}$$

taking into account the periodicity of the flow and conditions (1.2) and (2.1). Here $\bar{V} = V_x - iV_y$ is the complex velocity, λ is the wavelength, l is the length of the wave profile, $z = x + iy$, and the positive direction of traversing the contour L is the direction for which the domain D_1 in Fig. 1 is on the left. The last equation describes the velocity field induced by a vortex surface with intensity γ. Hence, in order to satisfy the condition (1.3) that no fluid flow across the interface, it suffices to take one half the sum of the boundary values of the flow velocity as the translation velocity of the interface, that is, $v = (v_1 + v_2)/2$. Then v is determined from the Sokhotskii–Plemelj formulas [10] by means of the following singular integral:

$$\bar{v}(s, t) = \frac{1}{2\lambda i}\int_0^{l(t)} \gamma(\sigma, t)\,\mathrm{ctg}\frac{\pi}{\lambda}\{\zeta(s, t) - \zeta(\sigma, t)\}\,d\sigma. \tag{2.3}$$

We note two consequences of the last equation:

$$\int_0^{l(t)} v_\tau(s, t)\,ds = 0, \tag{2.4}$$

$$\int_0^{l(t)} v_\nu(s, t)\,ds = 0, \tag{2.5}$$

where v_τ and v_ν are, respectively, the components of the velocity $\mathbf{v}$ tangent and normal to the wave profile (see Fig. 1).

In a coordinate system associated with an arbitrary point of the line L and moving with velocity $\mathbf{v}$, the Cauchy–Lagrange integral of the equations of motion of the fluid has the form

$$\frac{p_n}{\rho_n} + \frac{\delta\Phi_n}{\delta t} + \frac{\mathbf{v}^2_{rn} - \mathbf{v}^2}{2} + g\eta = F_n(t), \quad n = 1,2. \tag{2.6}$$

Here ρ_n is the density of the fluid; the differentiation $\delta/\delta t$ is performed in the moving coordinate system, so that $v = (\delta\xi/\delta t,\ \delta\eta/\delta t)$; Φ_n and $\mathbf{v}_{rn} = \mathbf{v}$ are the limiting values of the velocity potential and relative fluid velocity, respectively, on approaching L from the domain D_n; g is the acceleration due to gravity; and F_n are arbitrary functions. Taking into account (2.1), (2.3), and the equations

$$\mathbf{v}_{r1} = -\frac{\gamma}{2}\tau, \quad \mathbf{v}_{r2} = \frac{\gamma}{2}\tau,$$

$$\Phi_n(s, t) = \Phi_n(0, t) + \int_0^s \left\{v_\tau + (-1)^n \frac{\gamma}{2}\right\} d\sigma, \quad n = 1,2, \tag{2.7}$$

where τ is a unit tangent to the line L (see Fig. 1), we can obtain from (2.6) the following expression for the pressure drop at the wave profile:

$$p_1(s) - p_2(s) = \frac{1}{2}\frac{\delta}{\delta t}\int_0^s \{(\rho_1 + \rho_2)\gamma + 2(\rho_2 - \rho_1)v_\tau\}\,d\sigma + \frac{\rho_2 - \rho_1}{2}\left(2g\eta + \frac{\gamma^2}{4} - \mathbf{v}^2\right) + \chi(t).$$

Here the initial reference point for the arc length moves with velocity $\mathbf{v}(0, t)$, and χ is some function depending on $F_1(t)$, $F_2(t)$, $\Phi_1(0, t)$, and $\Phi_2(0, t)$. Eliminating the function χ in the last equation and introducing the dimensionless parameter

$$R = (\rho_2 - \rho_1)/(\rho_2 + \rho_1), \tag{2.8}$$

we write condition (1.4) in the following form:

$$\frac{\delta}{\delta t}\int_0^s (\gamma + 2Rv_\tau)\,d\sigma = \left\{\frac{2\mu k}{\rho_1 + \rho_2} - R\left(2g\eta + \frac{\gamma^2}{4} - \mathbf{v}^2\right)\right\}\Bigg|_{\sigma=0}^{\sigma=s}. \tag{2.9}$$

Condition (1.2) leads to the following relation:

$$\int_0^{l(t)} \gamma(s, t)\, ds = 2v_\infty \lambda. \tag{2.10}$$

Thus, the boundary conditions (1.2)-(1.4) are represented in the form of Eqs. (2.3), (2.9), and (2.10). However, the motion of the wave profile is determined only by Eqs. (2.3) and (2.9), because condition (2.10) is a consequence of them.

3. For the purpose of simplifying the system (2.3) and (2.9) we go from the Eulerian arc length $s \in [0, l(t)]$ to the Lagrangian variable $a \in [-\pi, \pi]$ with the following correspondence in time between the points of the wave profile. The point $\zeta(a, t) = \zeta(s(a, t), t)$ is translated in time dt to the point $\zeta(a, t+dt) = \zeta(a, t) + v(s(a, t), t)dt$. As the variable a we can take, for example, the arc length $2\pi s/l$ at the time $t = 0$. We introduce the function

$$\Gamma(a, t) = \gamma(s(a, t), t)|\zeta_a(a, t)|$$

and switch to dimensionless linear quantities, taking $\lambda/2\pi$ as the unit of length. Then, the system (2.3) and (2.9) can be written as follows:

$$\bar{\zeta}_t(a, t) = \frac{1}{4\pi i} \int_{-\pi}^{\pi} \Gamma(\alpha, t) \operatorname{ctg} \frac{\zeta(a, t) - \zeta(\alpha, t)}{2}\, d\alpha, \tag{3.1}$$

$$\Gamma_t(a, t) + R \int_{-\pi}^{\pi} \Gamma_t(\alpha, t) K(a, \alpha, t)\, d\alpha = H(a, t), \tag{3.2}$$

where

$$K(a, \alpha, t) = \frac{1}{2\pi} Im\left\{\zeta_a(a, t) \operatorname{ctg} \frac{\zeta(a, t) - \zeta(\alpha, t)}{2}\right\}, \tag{3.3}$$

$$H(a, t) = \frac{R}{2\pi} Im\left\{\zeta_a(a, t) \int_{-\pi}^{\pi} \Gamma(\alpha, t) \frac{\zeta_t(a, t) - \zeta_t(\alpha, t)}{1 - \cos[\zeta(a, t) - \zeta(\alpha, t)]}\, d\alpha\right\} + \frac{\partial}{\partial a}\left\{\frac{2\mu k}{\rho_1 + \rho_2} - R\left(2g\eta + \frac{1}{4}\frac{\Gamma^2}{|\zeta_a|^2}\right)\right\}. \tag{3.4}$$

The initial condition (1.5) can be brought to the form

$$\Gamma(a, 0) = \Gamma_0(a), \quad \zeta(a, 0) = \zeta_0(a), \tag{3.5}$$

where Γ_0 and ζ_0 are given functions. Thus, the initial problem (1.1)-(1.5) reduces to the Cauchy problem with the initial data (3.5) for the system of equations (3.1)-(3.4). A similar system was given by Birkhoff [11] for the case $\mu = 0$.

Let us point out the following invariants of the system (3.1)-(3.4):

$$I_\gamma = \int_{-\pi}^{\pi} \Gamma(a, t)\, da = 4\pi v_\infty, \quad I_\tau - iI_\nu = \int_{-\pi}^{\pi} \zeta_a(a, t)\, \bar{\zeta}_t(a, t)\, da = 0, \tag{3.6}$$

where I_γ, I_τ, and I_ν denote the left sides of Eqs. (2.10), (2.4), and (2.5), respectively.

4. In a neighborhood of the time $t = 0$ let the functions $\partial\Gamma/\partial a$ and $\partial^3\zeta/\partial a^3$ exist and be Holder continuous with respect to a [otherwise, the system (3.1)-(3.4) would lose its meaning].

Then, the derivatives Γ_t and ζ_{tt} exist and are unique. In fact, let us find the velocity ζ_t from (3.1) and substitute it into the right side of (3.2). In the integral equation obtained for the function Γ_t the kernel K is Fredholm if it is defined on the line $a = \alpha$ as follows:

$$K(a, a, t) = \lim_{\alpha \to a} K(a, \alpha, t) = k(a, t)|\zeta_a(a, t)|.$$

Sukharevskii [12] has shown that the eigenvalues of a Fredholm integral equation of the second kind with kernal K lie outside the interval [−1, 1]; but it follows from (2.8) that $|R| \leq 1$, and therefore, for all R the function Γ_t is uniquely determined from Eq. (3.2). The Holder continuity of this function enables us to determine the acceleration ζ_{tt} from the expression

$$\bar{\zeta}_{tt}(a) = \frac{1}{4\pi i} \int_{-\pi}^{\pi} \left\{\Gamma_t(\alpha) \operatorname{ctg} \frac{\zeta(a) - \zeta(\alpha)}{2} - \Gamma(\alpha) \frac{\zeta_t(a) - \zeta_t(\alpha)}{1 - \cos[\zeta(a) - \zeta(\alpha)]}\right\} d\alpha, \tag{4.1}$$

obtained by differentiating (3.1) with respect to t.

Similarly we can show that if the functions $\partial^2\Gamma/\partial a^2$ and $\partial^4\zeta/\partial a^4$ exist and are Holder continuous with respect to a, then the derivatives Γ_{tt} and ζ_{ttt} exist and are unique. We obtain thereby the following scheme for determining the time derivatives of the functions Γ and ζ:

$$\zeta,\Gamma \to \zeta_t \to \Gamma_t \to \zeta_{tt} \to \Gamma_{tt} \to \zeta_{ttt}.$$

This recurrence scheme can be continued indefinitely, if in a neighborhood of the time in question the functions Γ and ζ have all derivatives with respect to a.

We note that by taking into account the identity (4.1) the system consisting of the equation

$$\Gamma_t + 2R\cdot \mathrm{Re}\left(\zeta_a\bar{\zeta}_{tt}\right) = \frac{\partial}{\partial a}\left\{\frac{2\mu k}{\rho_1+\rho_2} - R\left(2g\eta + \frac{1}{4}\frac{\Gamma^2}{|\zeta_a|^2}\right)\right\}$$

and Eq. (3.1) is equivalent to the system (3.1)-(3.4), but it has a more compact form.

5. Let us represent the complex potential of the flow $W=\Phi=i\Psi$ in the following form:

$$W(z,\ t)=(-1)^n v_\infty z+w(z,\ t),$$

where $n=1$ for $z \in D_1$, $n=2$ for $z \in D_2$, and $w\to 0$ for $|y|\to\infty$. Then the right sides of Eqs. (2.6) take the form $F_n = c_n + v_\infty^2/2$, where

$$c_1 = \lim_{y\to+\infty}\left\{\frac{1}{\rho_1}p(x,y,t)+gy\right\},$$

$$c_2 = \lim_{y\to-\infty}\left\{\frac{1}{\rho_2}p(x,y,t)+gy\right\}.$$

The quantities c_1 and c_2 depend only on time, and one of them is arbitrary. On the other hand, it can be shown that as one gets more distant from the interface the perturbed potential w damps out exponentially, and the integrals

$$\int_{M_n} w\,dz,\quad n=1,2,$$

where M_n is the contour ABP_nQ_nA (see Fig. 1), exist and are equal to zero. Hence, we see that

$$\int_{AB} W_n dz = (-1)^n v_\infty \lambda[\zeta(0)+\zeta(l)]/2,\quad n=1,2.$$

Here $W_n = \Phi_n + i\Psi_n$ is the limiting value of the potential W when L is approached from the domain D_n. After substituting expression (2.7) for Φ_n and the expression

$$\Psi_n(s,t) = \Psi_n(0,t) - \int_0^s v_\nu(\sigma,t)\,d\sigma,\quad n=1,2$$

into the last equation, we can determine Φ_n at the point $s=0$ up to a constant as follows:

$$\Phi_n(0,t) = \frac{1}{\lambda}\int_0^l\left\{\left[v_\tau + (-1)^n\frac{\gamma}{2}\right]\xi + v_\nu\eta\right\}ds,\quad n=1,2. \tag{5.1}$$

Now for the limiting values of the pressure on the wave profile we have from (2.6) the expression

$$\frac{p_n}{\rho_n} = \frac{v^2-\gamma^2/4}{2} - g\eta - \frac{\delta}{\delta t}\Phi_n(s,t) + c_n + \frac{v_\infty^2}{2},\quad n=1,2, \tag{5.2}$$

in which Φ_n is determined according to Eqs. (2.7) and 5.1).

6. Let the functions Γ and ζ be known at the time t. We look for the values of these functions at time $t+\Delta t$ according to the Taylor series

$$\Gamma(a,\ t+\Delta t)= \Gamma(a,\ t)+\Gamma_t(a,\ t)\Delta t+\Gamma_{tt}(a,\ t)(\Delta t)^2/2, \tag{6.1}$$

$$\zeta(a,\ t+\Delta t)=\zeta(a,\ t)+\zeta_t(a,\ t)\Delta t+\zeta_{tt}(a,\ t)(\Delta t)^2/2+\zeta_{ttt}(a,\ t)(\Delta t)^3/6. \tag{6.2}$$

The problem of calculating the time derivatives of the functions Γ and ζ ultimately reduces to the problem of numerical integration and differentiation. In order to solve this problem we partition the range of variation $[-\pi,\ \pi]$ of the Lagrangian variable into N intervals of equal length $2\pi/N$, and henceforth, we shall operate only with the values of the functions at the $N+1$ partition points. For the numerical integration the seventh-order Newton–Cotes integration formula is used. If the integral is singular, the singularity is isolated. Numerical differentiation with respect to the variable a is performed by a sixth-order difference-free method, based on approximating the function by a sixth-degree Legendre polynomial. The integral equation (3.2) reduces to a system of N linear, algebraic equations for the values of the function Γ_t at the partition points. This system is solved by using the Gauss–Jordan method to invert the matrix.

It is known that the surface of the contact discontinuity in a fluid is unstable in the sense that the initial perturbations grow faster, the smaller their wavelength [13]. To suppress this short-wavelength instability, it is necessary to introduce a smoothing procedure into the calculational scheme [14-16].

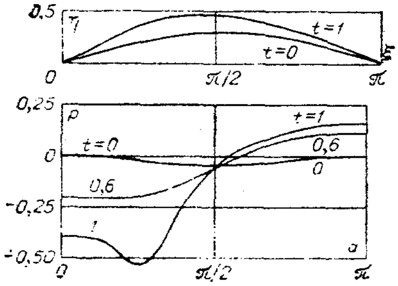

Fig. 2

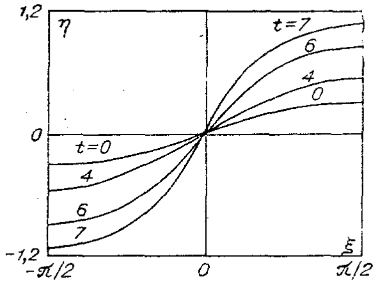

Fig. 3

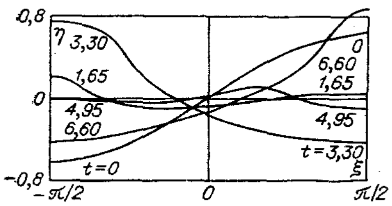

Fig. 4

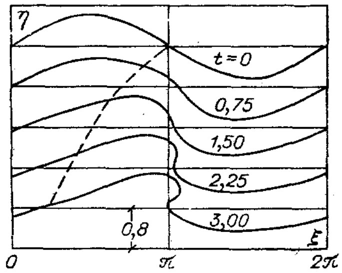

Fig. 5

In the present algorithm the smoothing is done at each step of the calculated values of Γ and ξ. Here, as the smoothed value of the function at some point we take the value at this point of the fifth-degree polynomial approximating the function at the given point and at the ten neighboring points (five to the left and five to the right) by the least-squares method.

To control the accuracy of the calculation we used the invariants (3.6) in the following manner. The calculation is stopped when at least one of the following inequalities is violated:

$$|I_\gamma - 4\pi v_\infty| < 0.1, \quad |I_\tau| < 0.1, \quad |I_\nu| < 0.1. \tag{6.3}$$

A program carrying out this algorithm was written in the language AL'GIBR for a BÉSM-6 computer. For N = 60 the calculation of one step according to Eqs. (6.1) and (6.2) took 14 sec of machine time. Ignoring the derivatives Γ_{tt} and ξ_{ttt}, it took 9 sec.

7. We note first that in all the variations considered below the unit of length is $\lambda/2\pi$, as in Sec. 3, so that the dimensionless wavelength is 2π.

1. Kelvin–Helmholtz Instability (Instability of the Line of Tangential Velocity Discontinuity in a Homogeneous Fluid). We take $\lambda/2\pi v_\infty$ as the unit of time and $\rho_1+\rho_2$ as the unit of density. The initial conditions are of the form $\xi(a,\ 0) = a + i(0.1)\pi \sin a$ and $\Gamma(0,\ a) = 2$. The calculation ran from t = 0 to t = 1 in steps of $\Delta t = 0.02$. In view of the symmetry of the flow with respect to the point $\xi(0,\ t) \equiv 0$ we can restrict ourselves to consideration of a half-wave. Its evolution during the indicated time is shown in Fig. 2, where points with the same Lagrangian coordinates are joined by straight line segments. In the same figure we give the distribution along the wave profile of the pressure p, calculated by Eq. (5.2) with $c_2(t) \equiv 0$. On a small portion of the profile we observe a drop in pressure and also an increase in the maximum and decrease in the minimum curvature. In a version of the calculation in which only the functions Γ_{tt} and ξ_{tt} are smoothed, this tendency is displayed more strongly. The indicated behavior of the profile curvature leads to a violation of conditions (6.3) when the calculation is continued.

The profile curvature and pressure at the point of the profile behave similarly when the calculation is done with the initial conditions $\xi(a,\ 0) = a + i(0.1)\pi \sin a$ and $\Gamma(a,\ 0) = \sin a$, modeling a vortex trace behind an oscillating profile [17].

2. Rayleigh–Taylor Instability (Instability in the Field of Gravity of the Interface between Two Fluids of Different Densities, when the Upper Fluid is Heavier than the Lower One). The unit of time is $(\lambda/2\pi g)^{1/2}$ and the parameters are $R = -0.1$ and $\mu/(\rho_1+\rho_2)\, g^{-1}\, \{\lambda/(2\pi)\}^{-2} = 0.01$. The initial conditions are of the form $\xi(a,\ 0) = a + i(0.1)\pi \sin a$ and $\Gamma(a,\ 0) = 0$. The calculation ran from t = 0 to t = 7 in steps of $\Delta t = 0.2$. The shape of the interface at various times is given in Fig. 3 (in view of the symmetry of the flow with respect to the line $x = -\pi/2$ and $x = \pi/2$ we restrict ourselves to consideration of a half-wave). Simultaneously with an increase in the amplitude of the wave we observe a growth in the curvature of the contour, which tends toward the formation of a discontinuity. The last circumstance leads to a violation of conditions (6.3) when the calculation is continued.

Before going on to consideration of the variations 3-8, we note that the initial conditions in these variations were taken from the linear theory [18].

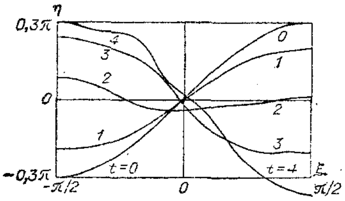

Fig. 6

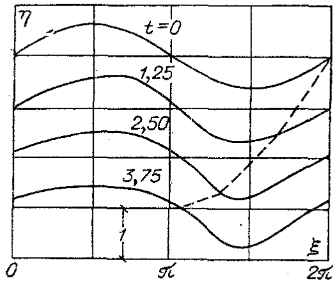

Fig. 7

3. Standing Gravity Waves on a Water–Air Interface ($R = 0.9975$). The unit of time is $(\lambda/2\pi g)^{1/2}$. The initial conditions are of the form $\zeta(a, 0) = a + i(0.2)\pi \sin a$ and $\Gamma(a, 0) = 0$. The calculation ran from $t = 0$ to $t = 7$ in steps of $\Delta t = 0.1$. In distinction to the previous versions the computation was stopped only because the given number of steps was exhausted. The evolution of the wave, shown in Fig. 4, is characterized by the following features (as in variation 2, we restrict ourselves to consideration of a half-wave). In the computation time the wave did not straighten out; fixed nodal lines are lacking; the maximum ordinate of the wave crest is greater than the minimum ordinate of the valley in absolute value; the maximum extension of the wave profile does not occur simultaneously with the maximum deviation of the profile from the unperturbed level ($y = 0$). These features of the standing wave agree with the conclusions of Sekerzh-Zen'kovich [19].

It should be mentioned that the computation of the evolution of the wave ignoring the functions Γ_{tt} and ζ_{ttt} in Eqs. (6.1) and (6.2) but with the same step size turned out to be unstable.

4. Progressive Gravity Wave on an Air–Water Interface ($R = 0.9975$) with No Wind ($v_\infty = 0$). The unit of time is $(\lambda/2\pi g)^{1/2}$. The initial conditions are of the form $\zeta(a, 0) = a + i(0.2)\pi \sin a$ and $\Gamma(a, 0) = (0.4)\pi \sin a$. The calculation ran until the given number of steps was exhausted from $t = 0$ to $t = 3$ in steps of $\Delta t = 0.025$, ignoring the derivatives Γ_{tt} and ζ_{ttt}. The shape of the wave at different times is given in Fig. 5 in the coordinate system moving in the positive x direction with the velocity of an infinitesimally small-amplitude wave. In this case the calculated points "slip" along the wave surface, so to speak, in the opposite direction. To illustrate this circumstance, points with the coordinate $a = 0$ are connected in the figure by dashed line segments. A remarkable feature of the wave is the formation of a crest hanging above the valley.

The commutation of the wave with allowance for Γ_{tt} and ζ_{ttt} and with the same step size ran stably up to $t = 2.25$.

5. Standing Capillary Wave on an Air–Water Interface ($R = 0.9975$). The unit of time is $(\lambda/2\pi)^{3/2}(\mu/(\rho_1 + \rho_2))^{-1/2}$. The initial conditions are of the form $\zeta(a, 0) = a + i(0.3)\pi \sin a$ and $\Gamma(a, 0) = 0$. The computation ran until the given number of steps was exhausted from $t = 0$ to $t = 4$ in steps of $\Delta t = 0.05$, ignoring the derivatives Γ_{tt} and ζ_{ttt}. The evolution of the wave (Fig. 6) is characterized by the following features (as in variation 2 we restrict ourselves to a half-wave). In the computation time the wave never straightens out; fixed nodal lines are lacking; and the minimum ordinate of the wave valley is greater than the maximum ordinate of the crest in absolute value. At the maximum extension of the wave profile there arise on it smaller waves. For example, at time $t = 4$ there are 10 inflection points on the wave profile.

The computation of the wave with allowance for the functions Γ_{tt} and ζ_{ttt} and with the same step size turned out to be unstable.

6. Progressive Capillary Wave on an Air–Water Interface ($R = 0.9975$) with No Wind ($v_\infty = 0$). The unit of time is $(\lambda/2\pi)^{3/2}(\mu/(\rho_1 + \rho_2))^{-1/2}$. The initial conditions are of the form $\zeta(a, 0) = a + i(0.2)\pi \sin a$ and $\Gamma(a, 0) = (0.4)\pi \sin a$. The computation ran until the given number of steps was exhausted from $t = 0$ to $t = 3.75$ in steps of $\Delta t = 0.025$, ignoring the derivatives Γ_{tt} and ζ_{ttt}. The shape of the waves for a sequence of times is given in Fig. 7 in the coordinate system moving in the positive x direction with the velocity of an infinitesimally small-amplitude wave. The dashed line segments join points with the coordinate $a = 0$. The calculated points, as in case 4, slip along the wave surface. A characteristic feature of the evolution of the wave is the fact that the wave crest tries to become gentler and the valley – steeper.

The computation of the wave with allowance for Γ_{tt} and ζ_{ttt} and with the same step size was unstable.

7. Wind-Driven Gravity Wave on an Air–Water Interface ($R = 0.9975$) with Froude Number $g\lambda/2\pi v_\infty^2 = 0.005$. The unit of time is $\lambda/2\pi v_\infty$. The initial conditions are of the form $\zeta(a, 0) = a + i(0.2)\pi \sin a$ and $\Gamma(a, 0) = 2 + (1.253)\sin a$. The computation ran from $t = 0$ to $t = 2.5$ in steps of $\Delta t = 0.05$. An interesting feature of the algorithm in this case is the concentration of the calculated points near the foot of the wave and the corresponding rarefaction in the neighborhood of the crest, while the shape of the wave is almost unchanged. This circumstance leads to violation of conditions (6.3) when the computation is continued.

8. Wind-Driven Capillary Wave on an Air–Water Interface ($R = 0.9975$) with the Dimensionless Parameter $(\mu/(\rho_1+\rho_2))(\lambda/2\pi)^{-1}v_\infty^{-2} = 1$. The unit of time is $\lambda/2\pi v_\infty$. The initial conditions are of the form $\zeta(a, 0) = a + i(0.2)\pi \sin a$ and $\Gamma(a, 0) = 2$. The computation ran from $t = 0$ to $t = 1.5$ in steps of $\Delta t = 0.025$ ignoring the functions Γ_{tt} and ζ_{ttt}. Analogous to the previous case there is a tendency for the calculated points to crowd together in the neighborhood of the leeward node of the wave and to become rarefied in the neighborhood of the windward node. This tendency leads to violation of conditions (6.3) when the computation is continued. Thus, for the numerical investigation of wind-driven waves changes in the algorithm are required.

In conclusion, the author would like to express his deep gratitude to D. N. Gorelov and A. V. Kazhikhov for valuable remarks.

LITERATURE CITED

1. L. V. Ovsyannikov, "Nonlocal Cauchy problems in hydrodynamics," in: International Congress of Mathematicians, Nice, 1970 [in Russian], Nauka, Moscow (1972).
2. V. I. Nalimov, "New model of the Cauchy–Poisson problem," in: Dynamics of Continuous Media [in Russian], No. 12, Nauka, Novosibirsk (1972).
3. R. M. Garipov, "Cavitation flow around an ellipsoid," in: Dynamics of Continuous Media [in Russian], No. 1, Nauka, Novosibirsk (1969).
4. T. D. Butler, Link method extensions," in: Lecture Notes in Physics, Vol. 8 (1971).
5. R. K.-C. Chan, R. L. Street, and J. E. Fromm, "The digital simulation of water waves – an evaluation of SUMMAC," in: Lecture Notes in Physics, Vol. 8 (1971).
6. C. Brennan, "Some numerical solutions of unsteady free surface wave problems using the Lagrangian description of the flow," in: Lecture Notes in Physics, Vol. 8 (1971).
7. A. K. Whitney, "The numerical solution of unsteady free surface flows by conformal mapping," in: Lecture Notes in Physics, Vol. 8 (1971).
8. P. N. Swartztrauber, "A numerical model of the unsteady free boundary of an ideal fluid," Quart. J. Appl. Math., 31, No. 2 (1973).
9. S. J. Zaroodny and M. D. Greenberg, "On a vortex sheet approach to the numerical calculation of water waves," J. Comput. Phys., 11, No. 3 (1973).
10. N. I. Muskhelishvili, Singular Integral Equations [in Russian], 3rd edition, Nauka, Moscow (1968).
11. G. Birkhoff, "Helmholtz and Taylor instability," in: Hydrodynamic Instability, Proceedings of Symposia in Applied Mathematics, Vol. 13 (1962).
12. I. V. Sukharevskii, "On the efficient calculation of the flow velocity and circulation in potential lattice flow," Tr. Khar'kov. Politekhn. In-ta, Ser. Inzh.-Fiz., 5, No. 1 (1955).
13. V. E. Zakharov and V. G. Kharitonov, "Instability of monochromatic waves on the surface of a liquid of arbitrary depth," Zh. Prikl. Mekh. i Tekh. Fiz., No. 5 (1970).
14. V. V. Rusanov, "Processing and analysis of computation results for multidimensional problems of aerohydrodynamics," in: Lecture Notes in Physics, Vol. 18 (1973).
15. A. A. Dorodnitsyn, "Plane problem of the unsteady motions of a heavy fluid," in: Applications of Function Theory to the Mechanics of Continuous Media, Transactions of the International Symposium, Tbilisi, September, 1963 [in Russian], Vol. 2, Nauka, Moscow (1965).
16. M. A. Lavrent'ev and B. V. Shabat, Problems of Hydrodynamics and Their Mathematical Models [in Russian], Nauka, Moscow (1973).
17. D. N. Gorelov and R. L. Kulyaev, "Nonlinear problem of nonstationary flow around a thin profile by an incompressible fluid," Izv. Akad. Nauk SSSR, Mekhan. Zhid. i Gaza, No. 6 (1971).
18. H. Lamb, Hydrodynamics, Dover (1932).
19. Ya. I. Sekerzh-Zen'kovich, "On the theory of finite-amplitude standing waves on the surface of a heavy fluid," Dokl. Akad. Nauk SSSR, 58, No. 4 (1947).